

ON THE FUNDAMENTAL CONSTITUTIVE
EQUATIONS IN ELECTROMAGNETIC THEORY

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ABSTRACT

The purpose of this paper is to investigate the absolute significance of the well-known constitutive relations in electromagnetic theory, and to see what simplifications are introduced by various assumptions as to the isotropic character of the space-time continuum. The "linear vector function" expressing the linear relationship between the two fundamental electromagnetic tensors $a_{rs} (\equiv \mathbf{B}, c\mathbf{E})$ and $b_{ns} (\equiv -\mathbf{D}, c\mathbf{H})$ is a tensor c_{ns}^{pq} of rank 4 which is covariant and contravariant of rank 2 respectively and is named the "structure-tensor." In order to express mathematically what is meant by the isotropic character of a medium, the notion of orthogonal ennuples is introduced (an orthogonal ennuple defined at a point in space-time enables us to turn the medium in the neighborhood of that point). A tensor is then determined by its invariants with respect to an orthogonal ennuple at the point at which the tensor is taken, rather than by its covariant or contravariant components with respect to the coordinate system of the manifold. The effect of various assumptions as to the isotropic character of space-time on the invariants of the "structure-tensor" c_{ns}^{pq} is investigated and it is found that if space is completely isotropic in the neighborhood of a point P in space-time, then the simple constitutive relations $\mathbf{B} = \mu\mathbf{H}$ and $\mathbf{D} = \epsilon\mathbf{E}$ (with μ and ϵ independent of one another) hold; and that to an observer in space-time this fact is indicated by a time-direction 4-vector issuing from P . Since μ and ϵ are distinct only in places where matter is present we can say that the distribution of matter in space-time is affected by the distribution of these time-direction vectors in space-time. The corresponding constitutive relations $\mathbf{B} = \mu\mathbf{H}$ and $\mathbf{D} = \epsilon\mathbf{E}$ for empty space (where $\mu\epsilon = 1$; Maxwell's relations) are found to hold in the neighborhood of points where these time-direction vectors are absent; i.e. such points are simply geometrical points. The analysis, as introduced in this paper, is a general one and is applied to elasticity theory. Thus the number of distinct coefficients of elasticity appearing in Hooke's law for a transversely-isotropic medium is found to be six instead of five, the reason for this disagreement being that the assumption as to the existence of an energy-density of deformation reduces the number from six to five.

AS IS well known¹ Maxwell's fundamental equations of electromagnetic theory can be very conveniently written in terms of two alternating covariant tensors of rank two (or six-vectors) in space of four dimensions. The first of these, a_{rs} consists of the magnetic induction vector $\mathbf{B}$ and the electric intensity vector $\mathbf{E}$ in such a way that

$$a_{23} = B_x; a_{31} = B_y; a_{12} = B_z; a_{14} = cE_x; a_{24} = cE_y; a_{34} = cE_z \quad (1.0)$$

(c denoting the velocity of light in vacuo) and one set of Maxwell's equations merely expresses the fact that the flux or surface integral of the six-vector

¹ A. Sommerfeld, Ann. d. Physik **32**, 749 (1910); **33** 649 (1910.) H. Bateman, Proc. London Math. Soc. **8**, 223 (1910); F. D. Murnaghan, Phys. Rev **17**, 74 (1921).

a_{rs} across any closed spread of two dimensions, imbedded in the given space-time of four dimensions (x, y, z, t), is zero. The second fundamental six-vector b_{rs} , consists of the electric displacement (or induction) vector $\mathbf{D}$ and the magnetic intensity vector $\mathbf{H}$ in such a way that

$$b_{23} = -D_x; b_{31} = -D_y; b_{12} = -D_z; b_{14} = cH_x; b_{24} = cH_y; b_{34} = cH_z \quad (1.1)$$

and the second set of Maxwell's equations states that the flux of this six-vector across any closed spread of two dimensions in the given space-time continuum is equal to the volume integral over the space of three dimensions (bounded by the spread of two dimensions) of the current-charge density four-vector. Now, in addition to Maxwell's equations (and quite distinct from them) are certain relations, connecting the four fundamental vectors, $\mathbf{B}$, $\mathbf{E}$, $\mathbf{D}$, $\mathbf{H}$ which are known as the constitutive equations. In space free from matter Maxwell made the hypothesis that these constitutive equations were

$$\mathbf{B} = \mathbf{H}; \mathbf{D} = \mathbf{E} \quad (1.2)$$

whilst in a material medium these equations were replaced by

$$\mathbf{B} = \mu \mathbf{H}; \mathbf{D} = \epsilon \mathbf{E} \quad (1.3)$$

where if the medium is isotropic μ and ϵ are two constants characteristic of the medium and known respectively, as its permeability and specific inductive capacity. If the medium is not isotropic μ and ϵ are linear vector functions so that the equation $\mathbf{B} = \mu \mathbf{H}$ for instance, is merely an abbreviation for three equations of the form

$$B_x = \mu_{11}H_x + \mu_{12}H_y + \mu_{13}H_z.$$

It is the purpose of the present paper to investigate the general linear relation connecting the two alternating covariant tensors a_{rs} and b_{rs} and to see what simplifications are introduced by various assumptions as to the isotropic character of the space-time continuum. The "linear vector function" expressing this relationship will be a tensor c_{rs}^{pq} of rank four which is covariant of rank two and contravariant of rank two and which may be assumed without any lack of generality, to be alternating in the contravariant and covariant labels separately. Thus we may write

$$b_{rs} = C_{rs}^{\alpha\beta} a_{\alpha\beta} \quad (1.4)$$

where the Greek labels α, β are summation labels and it is supposed that the summation runs over the six pairs 23, 31, 12, 14, 24, 34 formed from the four numbers 1, 2, 3, 4, ($x = x^1, y = x^2, z = x^3, t = x^4$). The most general situation would involve, then, $6 \times 6 = 36$ coefficients c_{rs}^{pq} . The tensor c_{rs}^{pq} which is the analogue of the linear vector function of elementary vector analysis will be called the structure-tensor.

In the section of our paper devoted to the investigation of the effect upon the structure-tensor of various assumptions as to the isotropic charac-

ter of space or space-time, we shall disregard the fact that in the application of our analysis with which we are primarily concerned, i.e., the electromagnetic theory, the tensors a_{rs} and b_{rs} are alternating. The reason for so doing is that a similar analysis is useful in other departments of theoretical physics, e.g. elasticity theory, where the tensors involved are not alternating. In elasticity theory they are symmetric, but it will be convenient not to make any special assumption as to the nature of these tensors until later. The structure-tensor c_{rs}^{pq} will accordingly involve $256 (=16 \times 16)!$ components and the Greek labels in the fundamental relation

$$b_{rs} = c_{rs}^{\alpha\beta} a_{\alpha\beta}$$

are supposed to run, independently, over the values 1, 2, 3, 4.

MATHEMATICAL DESCRIPTION OF ISOTROPY²

In order to express most conveniently what is meant by the isotropic character of a medium we shall suppose an orthogonal ennuple³ (ennuple = n^{ple} ; $n=4$) attached to each point of space-time. Denoting the four vectors of this ennuple by ${}_1\lambda, {}_2\lambda, {}_3\lambda, {}_4\lambda$ any one, ${}_r\lambda$ say, of these will be a contravariant vector whose components are ${}_r\lambda^p$ ($p=1, 2, 3, 4$) and we shall denote the covariant representation of this vector by ${}_r\lambda_p$. Hence we have the relations defining orthogonality

$${}_r\lambda^{\alpha s} \lambda_{\alpha} = \epsilon_r^s \quad (1.5)$$

where the ϵ_r^s are a set of invariants or scalar quantities having the values of the components of the unit tensor δ_s^r ; i.e. $\epsilon_r^s = 0$ if $s \neq r$ and 1 if $s = r$. By means of the tensors ${}_r\lambda$ and ${}_r\lambda$ any tensor may be described in terms of its invariants relative to the ennuple.⁴ For a tensor $A_{s_1 \dots s_k}^{r_1 \dots r_j}$ these are

² T. Levi-Civita, "Absolute Calculus" Ch. 10 sec. 2.

³ By an orthogonal ennuple we mean a pyramid (a generalization of the notion of the trihedron) whose edges are mutually orthogonal. The purpose of fixing a set of four mutually orthogonal directions at every point of our space-time is to provide a mechanism for turning space. Turning the coordinate system simply changes the numbers, for example, which determine a point in space-time.

When we say that any one of the edges of an orthogonal ennuple, the r th say, is presented by a contravariant tensor of rank one ${}_r\lambda^p$ ($p=1, 2, 3, 4$), we may consider these four components to be the direction cosines of the r th direction with respect to the coordinate system (x) of the space-time continuum. Thus, for example, let us consider a space of three dimensions in which everything is referred to a set of rectangular Cartesian axes and suppose that at any point P a rectangular frame is formed by the tangents (directed in the sense of increasing variables) to the space-polar coordinate curves r =variable, θ =variable and ϕ =variable passing through P . Then the tangent line to the θ =variable curve, for example, is described in the Cartesian coordinate system of our space by three components:

$${}_r\lambda^1 = \cos \theta \cdot \cos \phi; \quad {}_r\lambda^2 = \cos \theta \cdot \sin \phi; \quad {}_r\lambda^3 = -\sin \theta.$$

⁴ For example suppose a space-polar trihedron to be fixed at a point P of space and a vector R determined at this point. Let us suppose that the vector R represents a generalized force. Then this force may be determined by its covariant components ($-\partial\psi/\partial r$, $-\partial\psi/\partial\theta$, $-\partial\psi/\partial\phi$) or by its projections on the directions belonging to the space-polar trihedron at P ,

$$i_{s_1 \dots s_k}^{r_1 \dots r_j} = A_{\sigma_1 \dots \sigma_k}^{\rho_1 \dots \rho_j} r_1 \lambda_{\rho_1} \dots r_j \lambda_{\rho_j} s_1 \lambda^{\sigma_1} \dots s_k \lambda^{\sigma_k} \quad (1.6)$$

from which it immediately follows that

$$A_{n_1 \dots n_k}^{m_1 \dots m_j} = i_{\beta_1 \dots \beta_k}^{\alpha_1 \dots \alpha_j} \alpha_1 \lambda^{m_1} \dots \alpha_j \lambda^{m_j} \beta_1 \lambda_{n_1} \dots \beta_k \lambda_{n_k} \quad (1.7)$$

(In deriving which, one uses the elementary theorem of algebra that since the two sets $r\lambda^s$ and $s\lambda_q$ are reciprocal; i.e. $r\lambda^{\alpha s}\lambda_\alpha = \epsilon_r^s$, it follows that $\alpha\lambda^{\alpha s}\lambda_s = \delta_\alpha^r$). It follows without difficulty that the tensor relation

$$b_{rs} = c_{rs}^{\alpha\beta} a_{\alpha\beta}$$

is equivalent to a similar relation

$$j_{rs} = K_{rs}^{\alpha\beta} i_{\alpha\beta} \quad (1.8)$$

between the univariants of the tensors involved and it is in terms of the invariants K_{rs}^{pq} of the structure-tensor c_{rs}^{pq} that we shall find it convenient to express the conditions of isotropy.

Suppose now, that at any point of our space-time we have a second ennuple $1\mu, 2\mu, 3\mu, 4\mu$ whose direction-vectors are connected with those of the first set by the relations

$$r\mu^p = s_r^\alpha \lambda^\alpha \quad (1.9)$$

where the scalar coefficients s_r^α are the elements of an orthogonal matrix. We shall say that the ennuple μ is derived from the ennuple λ by a *rotation* if the determinant of the matrix of the s_r^α is 1 and by a *reflexion* if the determinant of this matrix is -1 . In fact the covariant presentations of the two ennuples are connected by the relation

$$i_{\mu k} = s'^{\alpha} i_{\alpha} \lambda_k \quad (2.0)$$

where s' is the conjugate matrix to s ($s'^p_q = s^q_p$) so that for the μ -ennuple to be also orthogonal it is necessary and sufficient that the product of the matrix s by its conjugate s' should be the unit matrix and this is the definition of an orthogonal matrix.

The relation between the invariants of any tensor relative to the λ and μ ennuples follows at once. Thus if we distinguish the invariants of a tensor relative to the μ ennuple from those relative to the λ ennuple by a superposed bar, we have, taking a covariant tensor a_{rs} of rank two as an illustration.

$$\bar{i}_{rs} = a_{\alpha\beta} r\mu^\alpha s_r^\beta = a_{\alpha\beta} s_r^\rho \cdot \rho\lambda^\alpha s_s^\sigma \cdot \sigma\lambda^\beta = i_{\rho\sigma} s_r^\rho s_s^\sigma. \quad (2.1)$$

We are primarily interested in the structure-tensor c_{rs}^{pq} and the relation between its invariants relative to the two ennuples is

$$\bar{K}_{rs}^{pq} = \bar{K}_{\alpha\beta\gamma\delta}^{\delta\epsilon\zeta\eta} s_r^\rho s_s^\sigma s_\epsilon^\alpha s_\zeta^\beta s_\eta^\gamma s_\delta^\beta \quad (2.2)$$

i.e. $(-\partial\psi/\partial r, -(1/r)(\partial\psi/\partial\theta), -(1/r \sin \theta)(\partial\psi/\partial\phi))$. These latter three quantities are invariants with regard to changes in the coordinate system but depend on the particular set of orthogonal directions at P .

The medium will be said to be isotropic (with regard to the structure-tensor c_{rs}^{pq}) relative to the given rotation or reflexion if the invariants $\bar{K}$ of the structure-tensor relative to the new ennuple are the same as its invariants relative to the original ennuple i.e. if

$$\bar{K}_{rs}^{pq} = K_{rs}^{pq}.$$

If the invariants $\bar{K}$ keep their numerical values but change their signs under reflexions the medium will be said to be relatively (as opposed to absolutely) isotropic with respect to the structure-tensor.

ISOTROPY WITH REGARD TO ROTATIONS IN THE COORDINATE PLANES

Now a rotation through a right angle in the 2-3 plane is described by the matrix

$$\begin{array}{c|cccc} & \begin{array}{c} \rightarrow \\ 1\lambda \end{array} & 2\lambda & 3\lambda & 4\lambda \\ \begin{array}{c} \downarrow \\ 1\mu \end{array} & 1 & 0 & 0 & 0 \\ \begin{array}{c} 2\mu \end{array} & 0 & 0 & 1 & 0 \\ \begin{array}{c} 3\mu \end{array} & 0 & -1 & 0 & 0 \\ \begin{array}{c} 4\mu \end{array} & 0 & 0 & 0 & 1 \end{array}$$

so that $s_2^3 = 1$, $s_1^1 = 1$, $s_3^2 = -1$, $s_4^4 = 1$ all the others being zero. Upon substituting these values in the 256 equations

$$K_{rs}^{pq} = K_{\alpha\beta\delta\epsilon}^{\delta\epsilon\ 1p\ 1q\ \alpha\ \beta} s_\epsilon^{\delta\epsilon} s_r^{\alpha\beta} s_s^{\delta\epsilon} \quad (2.3)$$

(which express the assumption that the invariants of the structure tensor are not affected by a rotation through a right angle in the 2-3 plane) we find that many of the invariants of the structure-tensor must be zero; thus if $r = 1$, $s = 1$, $p = 3$, $q = 1$, the only term in the quadruple summation which is different from zero is that for which $\alpha = 1$, $\beta = 1$, $\delta = 2$, $\epsilon = 1$ whence $K_{11}^{31} = -K_{11}^{21}$; similarly $K_{11}^{21} = K_{11}^{31}$ and combining these results we find that $K_{11}^{31} = K_{11}^{21} = 0$. Proceeding in this manner we find that out of the 256 invariants 128 are zero whilst there are 58 relations amongst the remaining 128 so that there are 70 distinct invariants. These may be succinctly displayed in the accompanying table where blanks indicate zero values, and where an abbreviation is introduced for K_{rs}^{pq} . Thus for instance, we write aa for K_{11}^{11} ; ig for K_{31}^{24} and so on. The key to this notation is given by

$$\begin{array}{|cccc|} \hline 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \\ \hline \end{array} \equiv \begin{array}{|cccc|} \hline a & h & g' & u \\ h' & b & f & v \\ g & f' & c & w \\ u' & v' & w' & d \\ \hline \end{array}$$

TABLE I.

		i															
		11	22	33	44	23	32	31	13	12	21	14	41	24	42	34	43
j	11	aa	ba	ba	da	fa	-fa					ua	u'a				
	22	ab	bb	bc	db	fb	-fc					ub	u'b				
	33	ab	bc	bb	db	fc	-fb					ub	u'b				
	44	ad	bd	bd	dd	fd	-fd					ud	u'd				
	23	af	bf	bf'	df	ff	ff'					uf	u'f				
	32	-af	-bf'	-bf	-df	ff'	ff					-uf	-u'f				
	31							gg	g'g	-g'h'	-gh'			vg	v'g	vh'	v'h'
	13							gg	g'g	-g'h	-gh			vg'	v'g'	vh	v'h
	12							gh	g'h	g'g'	gg'			vh	v'h	-vg'	-v'g'
	21							gh'	g'h'	g'g	gg			vh'	v'h'	-vg	-v'g
	14	au	bu	bu	du	fu	-fu					uu	uu'				
	41	au'	bu'	bu'	du'	fu'	-fu'					uu'	uu				
	24							gv	g'v	g'w	gw			vv	v'v	-vw	-v'w
	42							gv'	g'v'	g'w'	gw'			vv'	v'v'	-vw'	-v'w'
	34							gw	g'w	-g'v	-gv			vw	v'w	vv	v'v
	43							gw'	g'w'	-g'v'	-gv'			vw'	v'w'	vv'	v'v'

In the application of this analysis to electromagnetic theory the table will have 6×6 instead of 16×16 constituents. For isotropy as regards rotation through a right angle in the (2, 3) plane it will read

TABLE IA.

		i					
		23	31	12	14	24	34
j	23	ff			uf		
	31		gg	-gh		vg	-vh
	12		gh	gg		vh	vg
	14	fu			uu		
	24		gv	-gw		vw	-vw
	34		gw	gv		vw	vv

so that sixteen of the invariants are zero and only twelve of the remaining 20 are distinct.

When, on the other hand, the structure-tensor is symmetric in both its covariant and contravariant labels (a state of affairs which will arise when both of the tensors between which it sets up a linear relationship are symmetric) the columns and rows must coincide with the corresponding columns and rows for which the order of the labels is altered so that instead of $100 = 10 \times 10$ ($10 = 4 + 6$) distinct constituents there remain only 28, the table reading as follows:

TABLE IB.

		i									
		11	22	33	44	23	31	12	14	24	34
j	11	aa	ba	ba	da				ua		
	22	ab	bb	bc	db				ub		
	33	ab	bc	bb	db	-fb			ub		
	44	ad	bd	bd	dd				ud		
	23		bf	-bf		ff					
	31						gg	-gh		vg	vh
	12						gh	gg		vh	-vg
	14	au	bu	bu	du				uu		
	24						gv	gw		vv	-vw
	34						gw	-gv		vw	vv

We shall now find the new relations between the 70 distinct invariants of Table I, introduced by a rotation through any angle θ (other than a right angle) in the 2-3 plane. This rotation is described by the matrix

$$\begin{array}{c|cccc} & \begin{array}{c} \rightarrow i \\ 1\lambda \quad 2\lambda \quad 3\lambda \quad 4\lambda \end{array} \\ \begin{array}{c} \downarrow j \\ 1\mu \\ 2\mu \\ 3\mu \\ 4\mu \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & \sin \theta & 0 \\ 0 & -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

so that $s_1^1 = s_4^4 = 1$; $s_2^2 = s_3^3 = \cos \theta$; $s_2^3 = \sin \theta$; $s_3^2 = -\sin \theta$.

Upon substituting these values in the equations (2.3) and using the results embodied in Table I, we find the following new relations:

$$(a) \quad 2(K_{22}^{22} - K_{33}^{22} - K_{23}^{23} - K_{32}^{23}) \sin \theta \cos \theta = (K_{23}^{22} + K_{32}^{22})$$

$$(b) \quad 2(K_{22}^{22} - K_{33}^{22} - K_{23}^{23} - K_{32}^{23}) \sin \theta \cos \theta = -(K_{23}^{22} + K_{32}^{22})$$

from which we have that $K_{22}^{22} = K_{33}^{22} + K_{23}^{23} + K_{32}^{23}$ and $K_{23}^{22} = -K_{32}^{22}$

$$(c) \quad (K_{22}^{22} - K_{33}^{22} - K_{23}^{23} - K_{32}^{23})(\sin^2 \theta - \cos^2 \theta) \sin \theta \cdot \cos \theta = 2K_{23}^{22} \sin^2 \theta$$

$$(d) \quad (K_{22}^{22} - K_{33}^{22} - K_{23}^{23} - K_{32}^{23})(\sin^2 \theta - \cos^2 \theta) \sin \theta \cdot \cos \theta = 2K_{32}^{22} \sin^2 \theta$$

so that $K_{23}^{22} = K_{32}^{22}$ which combined with $K_{23}^{22} = -K_{32}^{22}$ gives $K_{23}^{22} = K_{32}^{22} = 0$

$$(e) \quad 2(K_{22}^{22} - K_{33}^{22} - K_{23}^{23} - K_{32}^{23}) \sin \theta \cdot \cos \theta = (K_{23}^{22} - K_{32}^{22}) + (K_{22}^{23} - K_{33}^{23})$$

or combined with the results of a , b , and c gives us

$$K_{22}^{23} = K_{33}^{23}.$$

Thus the new relations between the invariants K_{rs}^{pq} introduced by this rotation through an angle θ (excluding 90°) are

$$K_{22}^{22} = K_{33}^{22} + K_{23}^{23} + K_{32}^{23}; \quad K_{23}^{22} = 0; \quad K_{32}^{22} = 0; \quad K_{22}^{23} = K_{33}^{23}.$$

If we include these relations in Table I we shall have the conditions on the invariants for isotropy with regard to a rotation through any angle in the 2-3 plane. There are thus $70 - 4 = 66$ distinct invariants.

Table IA remains unchanged, whilst Table IB (since $K_{23}^{22} = 0$ and $K_{22}^{22} = K_{33}^{22} + 2K_{23}^{23}$) contains only $28 - 2 = 26$ distinct invariants, when we include the rotation through an angle θ .⁵

⁵ We remark that Table IB (disregarding the coefficients involving the 4th index) applies to the case of a tetragonal crystal with one of the edges an axis of symmetry in the sense that the other two edges of the crystal issuing from P and all rays (in the plane of these two edges) which may be made to coincide with either of these edges by a 90° rotation, are equivalent.

If we wish to find the conditions on the invariants for isotropy with regard to a rotation through any angle θ in some other coordinate plane, let us say the 1-2 plane, we may do so by inspection of Table I. We simply replace 23 by 12 and 32 by 21, whilst 1 is replaced by 3; for 3 stands in the same position with respect to 12, as 1 does to 23. For example, the condition $K_{11}^{32} = -K_{11}^{23}$ of table 1 is replaced by the relation $K_{33}^{21} = -K_{33}^{12}$. Similarly, instead of the relations

$$K_{22}^{22} = K_{33}^{22} + K_{23}^{23} + K_{32}^{23}; \quad K_{23}^{23} = K_{32}^{22} = 0; \quad K_{22}^{23} = K_{33}^{23}$$

we have

$$K_{11}^{11} = K_{22}^{11} + K_{12}^{12} + K_{21}^{12}; \quad K_{12}^{11} = K_{21}^{11} = 0; \quad K_{11}^{12} = K_{22}^{12}$$

In this way we may construct a table of invariants such that there will be structural symmetry with regard to any rotation θ in the 1-2 plane.

ISOTROPY WITH REGARD TO COMBINED ROTATIONS IN PLANES 2-3 AND 1-2.

If in addition to symmetry of structure with regard to a rotation through any angle in the 2-3 plane there is structural symmetry with regard to a similar rotation in the 1-2 plane we find without much fresh calculation the table of invariants for structural symmetry in the 2-3 and 1-2 planes. For example the relation $K_{11}^{12} = 0$ of Table I becomes $K_{33}^{23} = 0$ in the table of invariants for the 1-2 plane, both relations holding for the table of invariants for structural symmetry with regard to combined rotations in the 2-3 and 1-2 planes. Proceeding in this way we find that 192 of the 256 invariants are zero whilst of the remaining 64 only 12 are distinct. The following Table II expresses these facts and the symmetry of it shows that there is structural symmetry with regard to combined rotations through any angle in each of the planes 23, 31 and 12.

TABLE II.

	i															
	11	22	33	44	23	32	31	13	12	21	14	41	24	42	34	43
j	11	aa	ab	ab	da											
22	ab	aa	ab	da												
33	ab	ab	aa	da												
44	ad	ad	ad	dd												
23					ff						uf		u'f			
32					ff'	ff					-uf	-u'f				
31							ff	ff'					uf	u'f		
13							ff'	ff					-uf	-u'f		
12									ff'	ff'					uf	u'f
21															-uf	-u'f
14					fu	-fu					uu	uu'				
41					fu'	-fu'					uu'	uu				
24							fu	-fu					uu	uu'		
42							fu'	-fu'					uu'	uu		
34									fu	-fu					uu	uu'
43									fu'	-fu'					uu'	uu

with $K_{11}^{11} = K_{22}^{11} + K_{23}^{23} + K_{32}^{23}$

If we include rotations through any angle θ , i.e. the material now possesses an axis of symmetry such that all rays at right angles to this axis are equivalent, the table corresponding to IB contains a less number of coefficients, there being nine in the first case and only six in the second.

Corresponding to Tables IA we have Table IIA with only four distinct invariants and 24, zero.

TABLE IIA.

		23	31	12	14	24	34
$j \rightarrow$	i						
23		ff			uf		
31			ff			uf	
12				ff			uf
14		fu			uu		
24			fu			uu	
34				fu			uu

Similarly, when the structure-tensor is symmetric in both its covariant and contravariant labels we have a Table IIB corresponding to Tables IB. Thus

TABLE IIB.

		11	22	33	44	23	31	12	14	24	34
$j \rightarrow$	i										
11		aa	ab	ab	da						
22		ab	aa	ab	da						
33		ab	ab	aa	da						
44		ad	ad	ad	dd						
23						ff					
31							ff				
12								ff			
14									uu		
24										uu	
34											uu

With $K_{11}^{11} = K_{22}^{11} + 2K_{23}^{23}$

there being 78 invariants out of the possible 100 zero, whilst of the remaining 22 only 6 are distinct.

In resumé, the three Tables II, IIA and IIB present the surviving invariants under the assumption that the invariants of the structure-tensor are not affected by a rotation through any angle in the 23, 31 and 12 planes.

We shall now consider the most general orthogonal rotation of the 123-space about the 4th direction. That is, we shall investigate the invariants for structural symmetry about the ${}_4\lambda$ -axis.

ISOTROPY WITH REGARD TO A GENERAL ROTATION ABOUT THE λ -AXIS

The general rotation of the 123-space about the 4th direction is given by the matrix

		${}_1\lambda$	${}_2\lambda$	${}_3\lambda$	${}_4\lambda$
${}_1\mu$		s_1^1	s_1^2	s_1^3	0
${}_2\mu$		s_2^1	s_2^2	s_2^3	0
${}_3\mu$		s_3^1	s_3^2	s_3^3	0
${}_4\mu$		0	0	0	1

Upon substituting these values in the fundamental equations

$$K_{rs}^{pq} = K_{\alpha\beta\gamma\delta}^{\delta\epsilon\zeta\eta} s_{\epsilon}^p s_{\zeta}^q s_{\gamma}^{\alpha} s_{\delta}^{\beta}$$

(using the conditions expressed in Table II) we find the following new relations

$$K_{23}^{14} = \Delta K_{23}^{14}; \quad K_{23}^{41} = \Delta K_{23}^{41}; \quad K_{14}^{23} = \Delta K_{14}^{23}; \quad K_{41}^{23} = \Delta K_{41}^{23}$$

where Δ is the determinant of the matrix of this rotation, namely

$$\Delta = \begin{vmatrix} s_1^1 & s_1^2 & s_1^3 \\ s_2^1 & s_2^2 & s_2^3 \\ s_3^1 & s_3^2 & s_3^3 \end{vmatrix}$$

and since our matrix is an orthogonal one, we have

$$\Delta^2 = 1 \quad \text{or} \quad \Delta = \pm 1.$$

If $\Delta = +1$ then the rotation characterizes a movement of a rigid body about a fixed point. If $\Delta = -1$ the change of directions involves reflexion together with rotation.

ISOTROPY WITH REGARD TO A GENERAL ROTATION ABOUT A POINT P

We shall now consider the effect on the invariants K_{rs}^{pq} , if in addition to structural symmetry with respect to a general rotation about the 4λ -direction there is symmetry of structure with regard to a rotation through any angle in, let us say, the 1-4 plane.

Without fresh calculations but simply by inspection of Table I, we see that corresponding to the following relations therein;— $K_{22}^{22} = K_{33}^{33}$; $K_{33}^{33} = K_{11}^{11}$; $K_{22}^{33} = K_{33}^{22}$ we have (for the 1-4 plane) respectively: $K_{11}^{11} = K_{44}^{44}$; $K_{44}^{44} = K_{22}^{22}$; $K_{11}^{44} = K_{44}^{11}$ which when combined with Table II, yield

$$(a) \quad K_{44}^{44} = K_{11}^{11}; \quad K_{11}^{44} = K_{44}^{11} = K_{22}^{22}.$$

Similarly, the relations $K_{31}^{31} = K_{21}^{21}$ and $K_{13}^{31} = K_{12}^{21}$ for the 2-3 plane become respectively, $K_{42}^{42} = K_{12}^{12}$ and $K_{24}^{42} = K_{21}^{21}$ for the 1-4 plane which when taken together with table 2, yield

$$(b) \quad K_{14}^{14} = K_{23}^{23} \quad \text{and} \quad K_{41}^{14} = K_{32}^{23}.$$

Finally, the relations $K_{31}^{24} = -K_{12}^{43}$, $K_{12}^{42} = -K_{21}^{43}$, $K_{24}^{31} = -K_{34}^{21}$ for the 2-3 plane become respectively $K_{42}^{13} = -K_{12}^{43}$, $K_{24}^{31} = -K_{12}^{34}$, $K_{13}^{42} = -K_{43}^{12}$ for the 1-4 plane which, when combined with Table II, yield

$$(c) \quad K_{23}^{14} = K_{14}^{23}; \quad K_{23}^{41} = -K_{14}^{23}; \quad K_{41}^{23} = -K_{14}^{23}.$$

If now we impose on the invariants K_{ns}^{pq} the conditions that they be unaffected by a general orthogonal rotation of the ennuple (as a whole) about P , the matrix of this rotation being

$$\begin{array}{c|cccc} & 1\lambda & 2\lambda & 3\lambda & 4\lambda \\ \hline 1\mu & s_1' & s_1^2 & s_1^3 & s_1^4 \\ 2\mu & s_2' & s_2^2 & s_2^3 & s_2^4 \\ 3\mu & s_3' & s_3^2 & s_3^3 & s_3^4 \\ 4\mu & s_4' & s_4^2 & s_4^3 & s_4^4 \end{array}$$

we find upon substitution in the fundamental equations $K_{ns}^{pq} = K_{\alpha\beta}^{\delta\epsilon} s'^p s'^q s_r^\alpha s_r^\beta$ taking into account the conditions expressed by Table II and (a), (b) and (c) above, that the new relation obtained is simple

$$K_{14}^{23} = \Delta \cdot K_{14}^{23}$$

where

$$\Delta = \begin{vmatrix} s_1^1 & s_1^2 & s_1^3 & s_1^4 \\ s_2^1 & s_2^2 & s_2^3 & s_2^4 \\ s_3^1 & s_3^2 & s_3^3 & s_3^4 \\ s_4^1 & s_4^2 & s_4^3 & s_4^4 \end{vmatrix} = \pm 1.$$

These results are best expressed in Tables III, IIIA and IIIB corresponding to Tables II, IIA and IIB. Thus

TABLE III.

j	i															
	11	22	33	44	23	32	31	13	12	21	14	41	24	42	34	43
11	aa	ab	ab	ab												
22	ab	aa	ab	ab												
33	ab	ab	aa	ab												
44	ab	ab	ab	aa												
23					$\overline{ff}$	$\overline{ff}'$					fu	$-fu$				
32					$\overline{ff}'$	$\overline{ff}$					$-fu$	fu				
31							$\overline{ff}$	$\overline{ff}'$					fu	$-fu$		
13							$\overline{ff}'$	$\overline{ff}$					$-fu$	fu		
12									$\overline{ff}$	$\overline{ff}'$					fu	$-fu$
21									$\overline{ff}'$	$\overline{ff}$					$-fu$	fu
14											$\overline{ff}$	$\overline{ff}'$				
41					fu	$-fu$					$\overline{ff}'$	$\overline{ff}$				
24					$-fu$	fu							$\overline{ff}$	$\overline{ff}'$		
42							fu	$-fu$					$\overline{ff}'$	$\overline{ff}$		
34									fu	$-fu$					$\overline{ff}'$	$\overline{ff}$
43									$-fu$	fu					$\overline{ff}$	$\overline{ff}'$

with $K_{11}^{11} = K_{22}^{11} + K_{23}^{23} + K_{32}^{23}$

so that there are only four distinct invariants remaining.

TABLE IIIA.

		233112142434					
↓ j	i						
	23	ff			fu		
	31		ff			fu	
	12			ff			fu
	14		fu		ff		
	24			fu		ff	
	34				fu		ff

there being only two distinct invariants K_{23}^{23} and K_{23}^{14} out of 12 different from zero.

TABLE IIIB.

		11223344233112142434									
↓ j	i										
	11	aa	ab	ab	ab						
	22	ab	aa	ab	ab						
	33	ab	ab	aa	ab						
	44	ab	ab	ab	aa						
	23					ff					
	31						ff				
	12							ff			
	14								ff		
	24									ff	
	34										ff

$$\text{with } K_{11}^{11} = K_{22}^{11} + 2K_{23}^{23}$$

so that as in table IIB, 78 of the invariants are zero, whilst only two of the remaining 22 are distinct.

APPLICATIONS TO ELECTROMAGNETIC THEORY

a. *Absolute isotropy with regard to a general rotation around the 4λ -direction.* This means that $\Delta = +1$. Then from Table IIA, we find the relations between the invariants j_{rs} and i_{rs} of the two alternating covariant electromagnetic tensors b_{rs} and a_{rs} respectively:

$$\begin{aligned} j_{23} &= \alpha i_{23} + \beta i_{14}; & j_{31} &= \alpha i_{31} + \beta i_{24}; & j_{12} &= \alpha i_{12} + \beta i_{34} & \text{where } \alpha &= K_{23}^{23}; \beta &= K_{23}^{14}; \\ j_{14} &= \lambda i_{23} + \gamma i_{14}; & j_{24} &= \lambda i_{31} + \gamma i_{24}; & j_{34} &= \lambda i_{12} + \gamma i_{34} & \lambda &= K_{14}^{23}; \gamma &= K_{14}^{14}. \end{aligned} \quad (2.4)$$

Equations (2.4) are relations between the invariants j_{rs} and i_{rs} of the two electromagnetic tensors b_{rs} and a_{rs} respectively and are valid for any Riemannian space of four dimensions R_4 . If, however, we wish to find the corresponding set of relations between the tensor components we must know definitely the ennuple with respect to which the invariants j_{rs} and i_{rs} are taken.

Suppose that the system of coordinates of our R_4 is an orthogonal-curvilinear one for which the fundamental quadratic form is

$$ds^2 = H_1^2(dx^1)^2 + H_2^2(dx^2)^2 + H_3^2(dx^3)^2 + H_4^2(dx^4)^2,$$

and furthermore that the ennuple at P coincides with this reference frame. Then, the relations between the invariants j_{rs} , i_{rs} and the corresponding tensors b_{rs} , a_{rs} are such that

$$j_{rs} = b_{rs}/H_r H_s; \quad i_{rs} = a_{rs}/H_r H_s$$

so that corresponding to equations (2.4) between the invariants j_{rs} and i_{rs} we have the following set of relations between the tensors b_{rs} and a_{rs} :

$$\begin{aligned} \frac{H_1}{H_2 H_3} b_{23} &= \frac{H_1}{H_2 H_3} \alpha a_{23} + \frac{\beta}{H_4} a_{14}; & \frac{1}{H_4} b_{14} &= \frac{H_1}{H_2 H_3} \lambda a_{23} + \frac{\gamma}{H_4} a_{14} \\ \frac{H_2}{H_3 H_1} b_{31} &= \frac{H_2}{H_3 H_1} \alpha a_{31} + \frac{\beta}{H_4} a_{24}; & \frac{1}{H_4} b_{24} &= \frac{H_2}{H_3 H_1} \lambda a_{31} + \frac{\gamma}{H_4} a_{24} \\ \frac{H_3}{H_1 H_2} b_{12} &= \frac{H_3}{H_1 H_2} \alpha a_{12} + \frac{\beta}{H_4} a_{34}; & \frac{1}{H_4} b_{34} &= \frac{H_3}{H_1 H_2} \lambda a_{23} + \frac{\gamma}{H_4} a_{34} \end{aligned} \quad (2.5)$$

We now recall that the tensor a_{rs} consists of the magnetic induction vector $\mathbf{B}$ and the electric intensity vector $\mathbf{E}$; whilst the other electromagnetic tensor b_{rs} consists of the electric displacement vector $\mathbf{D}$ and of the magnetic force vector $\mathbf{H}$. Furthermore, let us suppose that $H_1 = H_2 = H_3 = 1$ and that $H_4 = ic$ i.e. our space-time is a Galilean one. Then equations (2.5) become simply

$$-\mathbf{D} = \alpha \mathbf{B} - i\beta \mathbf{E} \quad \text{and} \quad \frac{1}{i} \mathbf{H} = \lambda \mathbf{B} + \frac{\nu}{i} \mathbf{E} \quad (2.6)$$

$$\text{or } \mathbf{D} = i\beta \mathbf{E} - \alpha \mathbf{B} \quad \text{and} \quad \mathbf{B} = \frac{1}{i\lambda} \mathbf{H} - \frac{\nu}{i\lambda} \mathbf{E}$$

and comparing with the relations $\mathbf{D} = \epsilon \mathbf{E} + \rho \mathbf{B}$; $\mathbf{B} = \mu \mathbf{H} + \delta \mathbf{E}$ we have that $\epsilon = i\beta$, $\mu = 1/i\lambda$, $\rho = -\alpha$ and $\delta = -\gamma/i\lambda$ or $\epsilon\mu = \beta/\lambda$ and $\rho/\delta = \mu\alpha/\gamma$

b. *Absolute isotropy with regard to a general rotation about P.* By definition $\bar{K}_{rs}^{pq} = K_{rs}^{pq}$ and $\Delta = +1$ so that from Table IIIA we have the following relations between the invariants j_{rs} and i_{rs}

$$\begin{aligned} j_{23} &= \alpha i_{23} + \beta i_{14}; & j_{31} &= \alpha i_{31} + \beta i_{24}; & j_{12} &= \alpha i_{12} + \beta i_{34} \\ j_{14} &= \beta i_{23} + \alpha i_{14}; & j_{24} &= \beta i_{31} + \alpha i_{24}; & j_{34} &= \beta i_{12} + \alpha i_{34} \end{aligned} \quad (2.7)$$

where we now have only two distinct invariants $\alpha = K_{23}^{23}$, and $\beta = K_{14}^{23}$. Corresponding to equations (2.6) we have

$$\mathbf{D} = i\beta \mathbf{E} - \alpha \mathbf{B} \quad \text{and} \quad \mathbf{B} = \frac{1}{i\beta} \mathbf{H} - \frac{\alpha}{i\beta} \mathbf{E} \quad (2.8)$$

$$\text{or } \mathbf{D} = \epsilon \mathbf{E} + \gamma \mathbf{B} \quad \text{and} \quad \mathbf{B} = \mu \mathbf{H} + \delta \mathbf{E}$$

where $\mu\epsilon = 1$ (Maxwell's relation for the aether), and $\rho/\delta = \mu$ or $1/\epsilon$.

c. *Relative isotropy with regard to a general rotation about the ${}_4\lambda$ -axis.* That is, $\bar{K}_{rs}^{pq} = -K_{rs}^{pq}$ with $\Delta = -1$. Then in Table IIA $K_{23}^{23} = K_{14}^{14} = 0$ whilst $K_{23}^{14} = -\Delta K_{23}^{14} = K_{23}^{14}$ and $K_{14}^{23} = -\Delta K_{14}^{23} = K_{14}^{23}$ so that the relations between the invariants j_{rs} and i_{rs} are

$$\begin{aligned} j_{23} &= \beta i_{14}; \quad j_{31} = \beta i_{24}; \quad j_{12} = \beta i_{34} \\ j_{14} &= \lambda i_{23}; \quad j_{24} = \lambda i_{31}; \quad j_{34} = \lambda i_{12} \end{aligned}$$

where β and λ are distinct invariants (2.9) and corresponding to equations 2.6 we have simply

$$D = i\beta E \quad \text{and} \quad B = \frac{1}{i\lambda} H \tag{3.0}$$

with $i\beta = \epsilon$ and $i\lambda = 1/\mu$; or $\epsilon\mu = \beta/\lambda$.

Equations (3.0) are precisely the ones we wished to investigate and we see that the conditions necessary for them to be valid are a relatively-isotropic medium round the time-direction vector orthogonal to the space R_3 (considered as a hypersurface in space-time).

The equations $B = \mu H$, $D = \epsilon E$ (where μ , ϵ are distinct) are valid wherever matter is present. Considered from the point of view that our space R_3 is a hypersurface in space-time R_4 , we may say that if matter is present at a point P in our space-time, this fact is made known to an observer (in space-time) by a time-direction ${}_4\lambda$ attached to this point and orthogonal to the hypersurface R_3 .

d. *Relative isotropy with regard to a general rotation about a point P.* Table IIIA with $\overline{K}_{rs}^{pq} = -K_{rs}^{pq}$ and $\Delta = -1$ applies to this case. Corresponding to equations (2.9) we have the following set:

$$\begin{aligned} j_{23} &= \beta i_{14}; \quad j_{31} = \beta i_{24}; \quad j_{12} = \beta i_{34} \\ j_{14} &= \beta i_{23}; \quad j_{24} = \beta i_{31}; \quad j_{34} = \beta i_{12} \end{aligned} \tag{3.1}$$

where $\beta = K_{14}^{23}$, which in the special case of coincidence between the ennuple at P and a Galilean reference frame attached to P , become

$$D = i\beta E \quad \text{and} \quad B = \frac{1}{i\beta} H \tag{3.2}$$

or

$$D = \epsilon E \quad \text{and} \quad B = \mu H$$

with $\epsilon\mu = 1$ (Maxwell's relation for empty space). We thus see that when the entire space-time is relatively isotropic, i.e. $B = \mu H$ and $D = \epsilon E$ with $\mu\epsilon = 1$, then the fact that *no matter* is present in the neighborhood of the point P (x, y, z, t) is made known to an observer (in space-time) by the absence of a time-direction vector at P , orthogonal to the hypersurface R_3 .

APPLICATIONS TO ELASTICITY THEORY

Hooke's law on the linear dependence of stress on strain when the latter is small, says that each of the six components of the stress at any point of a body is a linear function of the six components of the strain at that point.

Thus at each point of our medium we are provided with two symmetric tensors, the stress tensor T_{rs} and the strain tensor e_{rs} . The linear relation between them is expressed by the equation

$$T_{rs} = c_{rs}^{\alpha\beta} e_{\alpha\beta} \quad (r, s = 1, 2, 3) \quad (3.3)$$

where $c_{rs}^{\alpha\beta}$ may be assumed, without any loss in generality, to be symmetric in the covariant and contravariant indices separately.

Corresponding to the tensor equation (3.2) we have an invariant equation

$$p_{rs} = K_{rs}^{\alpha\beta} q_{\alpha\beta} \quad (3.4)$$

where p_{rs} , $K_{rs}^{\alpha\beta}$, q_{rs} are the invariants of T_{rs} , $C_{rs}^{\alpha\beta}$, e_{rs} respectively, with respect to an orthogonal ennuple attached to the point P at which these tensors are defined.

ABSOLUTE ISOTROPY ROUND AN AXIS

If the medium possesses structural symmetry round an axis we have what is known as *transverse-isotropy*. We find from Table IIB that in this case there are only six distinct invariants. One finds, however, in the literature on the subject that only five independent coefficients are involved in Hooke's law for transverse-isotropy. This disagreement is due to the fact that we have not assumed the existence of an energy-density of deformation, which assumption would lead us to the equation

$$K_{rs}^{pq} = K_{pq}^{rs}$$

and therefore would reduce the number of distinct coefficients of elasticity from six to five. However, we see that as far as Hooke's law is concerned we have six distinct coefficients.

ABSOLUTE ISOTROPY WITH REGARD TO THE ENTIRE SPACE

When the medium possesses structural symmetry round all lines, i.e. there is complete or absolute isotropy, then from Table IIIB (which applies in this case) we see that Hooke's law contains only two independent coefficients of elasticity K_{11}^{11} and K_{23}^{23} which in the familiar notation of Lamé are identified with λ and μ .

BIOGRAPHY

Carl Kaplan, son of Henry Kaplan and Rosa Löwy Kaplan was born in Tapolcza, Hungary on April 17, 1904. His early education was received in the public schools of Baltimore and at the Baltimore Polytechnic Institute. He received his B.S. degree in Chemical Engineering at Johns Hopkins University in 1926 and his M.A. degree in Physics in 1928 at the same University. He was a Junior instructor in Mathematics in the years 1928-1929-1930. During the years 1927-28-29-30 he pursued graduate courses in Physics under Professors K. F. Herzfeld, A. H. Pfund, R. W. Wood and in Mathematics under Professors F. D. Murnaghan and A. Cohen.

